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Classical entanglement of twisted random light propagating through atmospheric turbulence [Invited]

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We examine the impact of the atmospheric turbulence on a recently discovered type of classical entanglement of partially coherent beams endowed with a twist phase. We derive a compact analytical expression for the Schmidt number of a bi-orthogonal decomposition of the Wigner function of a twisted Gaussian Schell-model (TGSM) beam propagating through the turbulent atmosphere. We elucidate conditions for a TGSM source to generate a strongly classically entangled paraxial field over a desired propagation distance in the turbulent atmosphere. Our results will find applications to free-space optical communications and motivate further research into classical entanglement with random light. © 2022 Optica Publishing Group

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1. INTRODUCTION

Virtually any viable free-space optical (FSO) communication protocol, whether classical or quantum, has to overcome the deleterious effects of the atmospheric turbulence on the light propagating from a source to a receiver [1]. In this context, it was demonstrated not too long ago [2,3] that reducing spatial coherence of the source can help mitigate the effects of turbulence to some extent. Most of the work in this direction to date, however, has been focused on employing statistically homogeneous light sources, such as a Gaussian Schell-model source [4,5]. More recently, the quest for applications of partially coherent light to free-space optical (FSO) communications has shifted to explore nonuniformly correlated optical sources [6–8]. In particular, it has been demonstrated lately [9] that any member of a certain class of partially coherent vortex fields with a structured cross-spectral density in a closed form derived in [10] is able to maintain both its vortex structure and spatial intensity distribution of light in the atmosphere under any turbulence conditions. The separability of the orbital angular momentum (OAM) of such beams, induced by their vortex structure, from their spatial distribution in the transverse plane and their robustness against the atmospheric turbulence, make these beams attractive candidates for FSO.

At the same time, there has been growing interest in exploring the application of classical entanglement between the polarization and spatial degrees of freedom of OAM carrying beams [11] to free-space classical [12] and quantum [13] communications. In this connection, a recent discovery of a link between classical entanglement and a nonlocal twist phase of partially coherent light [14] may offer new perspectives for FSO by combining the advantages of partial coherence for turbulence mitigation and classical entanglement associated with the twist phase. The twist phase of random light was first examined in connection with free-space propagation of twisted Gaussian Schell-model (TGSM) light beams [15]. The TGSM beams have since been experimentally generated [16,17], shown to arise in nonlinear media [18,19], employed to study information transfer through random media [20,21], and to improve optical image resolution [22,23].

In the context of free-space communications with TGSM beams, a natural question arises: How does the atmospheric turbulence affect the twist-induced entanglement of such beams? A related practical issue is concerned with the ability to structure a TGSM source to minimize the adverse affects of turbulence on their entanglement over a specified range of propagation distances. We stress that the TGSM beams can only be attractive for entanglement-reliant FSO protocols if the twist-induced entanglement can persist over sufficiently long propagation distances under any turbulence conditions.

In this work, we examine the effects of the atmospheric turbulence on classical entanglement of a TGSM source. We derive an expression for the Wigner function of a TGSM beam generated by the source at any distance from the source and evaluate the Schmidt number of its bi-orthogonal decomposition into a set of modes. We establish conditions for a TGSM source to generate a strongly classically entangled beam over a desired stretch of the turbulent atmosphere.

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The work is organized as follows. In the next section, we derive an analytical expression for the Wigner function of the optical field of a TGSM beam propagating in the turbulent atmosphere and evaluate the Schmidt number of a decomposition of the Wigner function into a set of bi-orthogonal modes. The following section is devoted to numerical evaluation of the Schmidt number under any turbulence conditions. Finally, we conclude with a brief summary of our results.

2. THEORETICAL FORMULATION

We start with an expression for the cross-spectral density $W(\mathbf{r}_1, \mathbf{r}_2, z)$ [5,24,25] of any paraxial optical field having propagated over a distance z in the turbulent atmosphere. The cross-spectral density of the field at a pair of points $\mathbf{r}_1$ and $\mathbf{r}_2$ in a plane transverse to the field propagation direction can be written in terms of the cross-spectral density $W_0(\mathbf{r}_1, \mathbf{r}_2)$ at the source as [1]

$$W(\mathbf{r}_1, \mathbf{r}_2, z) = \left(\frac{k_0}{2\pi z}\right)^2 \int \mathrm{d}\mathbf{r}'_1 \int \mathrm{d}\mathbf{r}'_2\, W_0(\mathbf{r}'_1, \mathbf{r}'_2) \times e^{ik_0\left(\mathbf{r}_2-\mathbf{r}'_2\right)^2/2z} e^{-ik_0\left(\mathbf{r}_1-\mathbf{r}'_1\right)^2/2z}\, \Gamma_m\left(|\mathbf{r}'_1 - \mathbf{r}'_2|\right), \quad (1)$$

where $k_0 = 2\pi/\lambda_0$ is a carrier wavenumber and a structure function Γ_m describes two point correlations of a random phase introduced into each statistical realization of the optical field by the medium turbulence. In a statistically homogeneous and isotropic turbulence, Γ_m can be approximated by a Gaussian of the form [1,26]

$$\Gamma_m(|\mathbf{r}_1 - \mathbf{r}_2|, z) = \exp\left[-\frac{|\mathbf{r}_1 - \mathbf{r}_2|^2}{2\sigma_m^2(z)}\right], \quad (2)$$

where a characteristic correlation width $\sigma_m(z)$ of the turbulence induced fluctuation reads

$$\sigma_m^2(z) = \frac{3}{2\pi^2 k_0^2 z \int_0^\infty \mathrm{d}\kappa\,\kappa^3 S_n(\kappa)}. \quad (3)$$

Although such a quadratic approximation for the structure function is not without pitfalls [27], it is expected to work well for low-coherence input sources for which the contributions to the tails of the integrand on the right-hand side of Eq. (1) are effectively cut off by short-range correlations at the source. This is precisely the type of source that is of interest in connection with the twist phase-induced classical entanglement [14].

We assume the spectrum $S_n(x)$ of the refractive index fluctuations entering the right-hand side of Eq. (3) to obey the Kolmogorov scaling with the von Kármán cutoff such that

$$S_n(\kappa) = 0.033 C_n^2 \frac{\exp(-\kappa^2/\kappa_>^2)}{(\kappa^2 + \kappa_<^2)^{11/6}}, \quad (4)$$

where C_n^2 is a so-called structure constant of the index fluctuations and $\kappa_> = 5.92/L_0$ and $\kappa_< = 2\pi/l_0$, with l_0 and L_0 being the inner and outer scales of the turbulence, respectively [1].

Let us now specialize to a twisted Gaussian Schell-model source whose cross-spectral density at a pair of points $\mathbf{r}_1$ and $\mathbf{r}_2$ in the source plane is given by [15]

$$W_0(\mathbf{r}_1, \mathbf{r}_2) \propto \exp\left(-\frac{\mathbf{r}_1^2 + \mathbf{r}_2^2}{4\sigma_I^2}\right) \exp\left[-\frac{(\mathbf{r}_1 - \mathbf{r}_2)^2}{2\sigma_c^2}\right] e^{iu(\mathbf{r}_1\times\mathbf{r}_2)_\perp}, \quad (5)$$

where $(\mathbf{r}_1 \times \mathbf{r}_2)_\perp = x_1 y_2 - x_2 y_1$ is a cross product of 2D vectors in the transverse plane of the source. Further, we introduced an rms width σ_I of the source intensity, source transverse coherence width σ_c, and a twist parameter u. Hereafter, we drop any immaterial constant factor.

To examine the evolution of classical entanglement of a TGSM beam in the turbulent atmosphere, we follow our recently introduced approach [14] by evaluating the Wigner function of the optical field of the beam. The Wigner function is defined by the expression

$$\mathcal{W}(\mathbf{k}, \mathbf{R}) = \int \mathrm{d}\mathbf{r}\, W(\mathbf{R} - \mathbf{r}/2, \mathbf{R} + \mathbf{r}/2) e^{-i\mathbf{k}\cdot\mathbf{r}}, \quad (6)$$

where

$$\mathbf{R} = (\mathbf{r}_1 + \mathbf{r}_2)/2, \quad \mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 \quad (7)$$

are the radius vectors of the "center-of-mass" and position difference of any two points $\mathbf{r}_1$ and $\mathbf{r}_2$ in the transverse plane of the beam. On substituting from Eq. (1) into (6) and using Eq. (2) through (4) we obtain, after a somewhat lengthy albeit straightforward calculation, the expression for the Wigner function of a TGSM beam as

$$\mathcal{W}(\mathbf{k}, \mathbf{R}) = \sigma_{\mathrm{eff}}^2(z) \mathcal{W}_+[k_y, X(z)] \mathcal{W}_-[k_x, Y(z)]. \quad (8)$$

Here

$$\mathcal{W}_+ \propto \exp\left[-\frac{X^2(z)}{2\sigma_I^2}\right] \exp\left\{-\frac{[k_y + uX(z)]^2 \sigma_{\mathrm{eff}}^2(z)}{2}\right\}, \quad (9)$$

and

$$\mathcal{W}_- \propto \exp\left[-\frac{Y^2(z)}{2\sigma_I^2}\right] \exp\left\{-\frac{[k_x - uY(z)]^2 \sigma_{\mathrm{eff}}^2(z)}{2}\right\}, \quad (10)$$

where we introduced the notations

$$X(z) = X - z k_x/k_0, \quad Y(z) = Y - z k_y/k_0, \quad (11)$$

as well as

$$\frac{1}{\sigma_{\mathrm{eff}}^2(z)} = \frac{1}{4\sigma_I^2} + \frac{1}{\sigma_c^2} + \frac{1}{\sigma_m^2(z)}. \quad (12)$$

At this point, let us briefly interpret the Wigner function evolution of a TGSM beam in the turbulent medium. We can infer from Eqs. (9) and (10) that in the absence of the twist phase, $u = 0$, the overall Wigner function can be rearranged into a product of two factors, each depending only on one of the two mutually orthogonal components of $\mathbf{k}$ and $\mathbf{R}$. Thus, the Wigner function is separable in Cartesian coordinates. Further, the evolution of the Wigner function distribution in physical space amounts to a shear: each coordinate is translated on propagation at a rate dependent on the corresponding component of the wave vector according to Eq. (11). At the same time, the Wigner distribution narrows in the $\mathbf{k}$-space, indicating coherence loss (decorrelation) caused by the medium fluctuations.

As the twist phase is imparted on the source, however, $\mathcal{W}$ becomes classically entangled with respect to pairs of variables, $(X(z), k_y)$ and $(Y(z), k_x)$, resulting in rather complicated dynamics. To quantify classical entanglement, we observe that the Wigner functions in Eqs. (9) and (10) have exactly the same functional form as those of a TGSM source derived in [14] if one replaces X and Y with $X(z)$ and $Y(z)$, respectively, as well as σ_c with an effective propagation dependent coherence length $\sigma_*(z)$ defined as

$$\frac{1}{\sigma_*^2(z)} = \frac{1}{\sigma_c^2} + \frac{1}{\sigma_m^2(z)}. \tag{13}$$

Next, we can introduce effective twist and coherence parameters by the expressions

$$t_{\rm eff}(z) = u\sigma_*^2(z), \quad \xi_{\rm eff}(z) = \sigma_*(z)/\sigma_I. \tag{14}$$

We can then follow the approach of Ref. [14] step by step and define a Schmidt number of a bi-orthogonal decomposition of the TGSM beam Wigner function in terms of the squares of the corresponding eigenvalues, $\nu_n = \lambda_{n\pm}^2$ as

$$K = \frac{\left(\sum_{n=0}^{\infty} \nu_n\right)^2}{\sum_{n=0}^{\infty} \nu_n^2}. \tag{15}$$

We note in passing that thereby defined Schmidt number is a classical analog of a degree of correlation of a biparticle entangled quantum wave function that was originally introduced in [28] motivated by probabilistic considerations.

Further, following exactly the same line of reasoning as in Ref. [14], we can read off an expression for the Schmidt number at any distance z within the turbulent medium as

$$K(z) = \frac{1 + 2t_{\rm eff}^2/\xi_{\rm eff}^2 + \xi_{\rm eff}^2/4}{1 + \xi_{\rm eff}^2/4}. \tag{16}$$

Next, on substituting from Eq. (14) into Eq. (16) and utilizing Eq. (13), we obtain, after simple algebra, the explicit expression for the Schmidt number in the form

$$K(z) = \frac{1 + 2t^2/\xi_c^2 + \xi_c^2/4 + \sigma_c^2/\sigma_m^2(z)}{1 + \xi_c^2/4 + \sigma_c^2/\sigma_m^2(z)}, \tag{17}$$

where $t = u\sigma_c^2$ and $\xi_c = \sigma_c/\sigma_I$ are dimensionless twist and coherence parameters of a TGSM source.

3. NUMERICAL RESULTS

At this stage, we interpret the just-derived expression for the Schmidt number. We can infer at once from Eq. (17) that (i) the second term in the numerator gives rise to strong entanglement of a nearly incoherent TGSM source, and (ii) the twist-induced entanglement is completely lost over long enough propagation distances over which the last term in the numerator and denominator dwarfs the rest, resulting in K asymptotically approaching unity. Thus, we can establish conditions for a TGSM source to generate strongly classically entangled beams over a given propagation distance L in the turbulent atmosphere. First, we stipulate that the source have low coherence, $\xi_c \ll 1$. Next, taking $t = 1$, we must ensure that the twist-induced entanglement generation dominates the turbulence-induced entanglement loss, at least, over the distance L so that

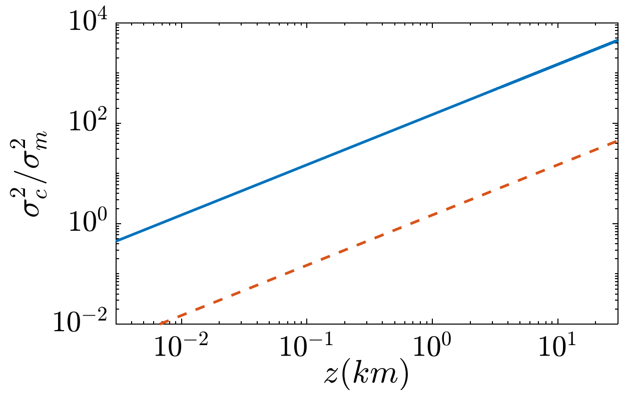

Fig. 1. σ_c^2/σ_m^2 as function of the propagation distance z (km) in a log-log scale for two TGSM sources with coherence widths: $\sigma_c = 1$ cm (dashed red line) and $\sigma_c = 10$ cm (solid blue line). The structure constant of refractive index fluctuations is taken to be $C_n^2 = 10^{-13}\ \mathrm{m}^{-2/3}$, corresponding to the strong fluctuation regime.

$$\sigma_c \ll (\sigma_I^2 L_T^3/L)^{1/4}, \tag{18}$$

where we introduced a length L_T associated with turbulence effects by the expression

$$L_T = \left[\frac{2\pi^2 k_0^2}{3}\int_0^\infty d\kappa\kappa^3 S_n(\kappa)\right]^{-1/3}. \tag{19}$$

Equations (17) and (18) are the key results of this work, as they make it possible to quantify the measure of twist-induced classical entanglement in FSO communications and indicate how to tailor a TGSM source to mitigate turbulence-induced entanglement degradation.

Under typical atmospheric conditions, the structure constant falls into the range: $10^{-17} \leq C_n^2 \leq 10^{-13}\ \mathrm{m}^{-2/3}$, with the lower bound corresponding to weak and the upper to strong fluctuation conditions. The turbulence length L_T is then in the range from around 60 cm in the strongly fluctuating regime to roughly 13 m in the weakly fluctuating one at the carrier wavelength $\lambda_0 = 0.5$ µm. To illustrate the effect of turbulence in the worst case scenario, we exhibit in Fig. 1 the ratio σ_c^2/σ_m^2 as a function of the propagation distance in the atmosphere in the log-log scale. We assume strong fluctuations ($C_n^2 = 10^{-13}\ \mathrm{m}^{-2/3}$) and display two cases: $\sigma_c = 1$ cm (dashed line) and $\sigma_c = 10$ cm (solid line). We can infer from the figure that even if the source remains nearly incoherent, $\sigma_c \ll \sigma_I$, turbulence effects can come into play over moderate propagation distances. For instance, for a nearly incoherent source with $\sigma_c = 10$ cm and $\lambda_0 = 0.5$ µm, such that a characteristic diffraction length of the beam generated by the source is of the order of $k_0\sigma_c^2 \sim 100$ km, turbulence effects become quite appreciable over a 10 km range.

Next, we examine the evolution of the Schmidt number of the Wigner function decomposition of a paraxial TGSM field into a bi-orthogonal set of modes in the turbulent atmosphere. In Fig. 2, we display K as function of z in the log-log scale for two sources: a nearly incoherent source with $\sigma_c = 1$ cm and $\sigma_I = 10$ cm (solid curve) and a rather incoherent one with the

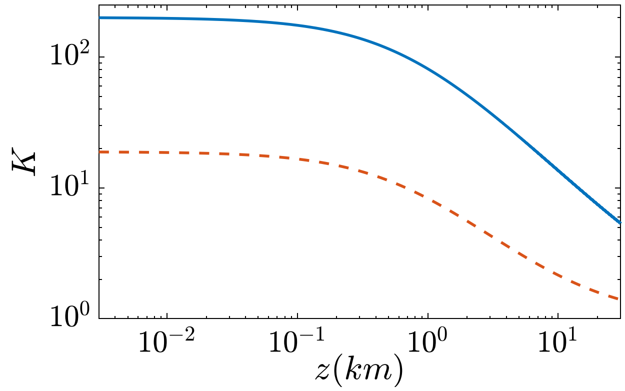

Fig. 2. Schmidt number K as function of the propagation distance z (km) in the log-log scale for two TGSM sources: $\sigma_I = 10$ cm and $\sigma_c = 1$ cm (solid blue curve) as well as $\sigma_I = 3$ cm and $\sigma_c = 10$ cm (dashed red curve). The structure constant of refractive index fluctuations is taken to be $C_n^2 = 10^{-13}$ m$^{-2/3}$, corresponding to the strong fluctuation regime.

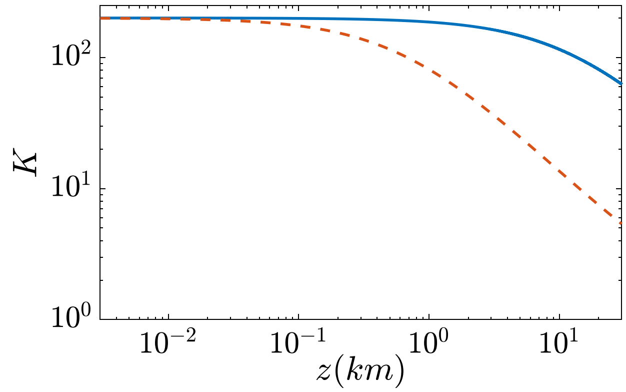

Fig. 3. Schmidt number K as function of the propagation distance z (km) in the log-log scale for a TGSM source with $\sigma_I = 10$ cm and $\sigma_c = 1$ cm for two atmospheric fluctuation regimes: $C_n^2 = 10^{-13}$ m$^{-2/3}$ (dashed red curve) and $C_n^2 = 5 \times 10^{-15}$ m$^{-2/3}$ (solid blue curve).

same coherence width and $\sigma_I = 3$ cm (dashed curve). We focus on the case of strong atmospheric fluctuations corresponding to $C_n^2 = 10^{-13}$ m$^{-2/3}$. We can conclude from the figure that entanglement remains nearly unaffected by the turbulence over the propagation distance of around 1 km and degrades precipitously afterward. We note that in both cases, the diffraction length is of the same order, $k_0\sigma_c^2 \sim 1$ km. We further note that our numerical results are consistent with Eq. (18), which limits the source coherence length to much less than 10 cm to ensure little entanglement degradation over a distance under 1 km.

Finally the strength of atmospheric fluctuations significantly affects the rate of entanglement loss, as is evidenced by Fig. 3, where we plotted K versus z in the log-log scale for a nearly incoherent source with $\sigma_c = 1$ cm and $\sigma_I = 10$ cm under moderate ($C_n^2 = 5 \times 10^{-15}$ m$^{-2/3}$) and strong ($C_n^2 = 10^{-13}$ m$^{-2/3}$) fluctuation conditions. The two cases correspond to solid and dashed curves, respectively. We observe in Fig. 3 that the twist-induced entanglement of the source is almost completely lost toward the end of a 30 km stretch of a strongly fluctuating atmosphere, while a substantial amount of entanglement is still maintained over the same interval under moderate fluctuation conditions.

4. SUMMARY

In summary, we have explored how the degree of classical entanglement of random light beams endowed with a twist phase is affected by beam propagation through turbulence. We have derived an elegant analytical expression for the Schmidt number of the bi-orthogonal decomposition of the Wigner function of a TGSM beam into a set of modes and traced the Schmidt number evolution as the beam propagates through the turbulent atmosphere. We have also established the condition for the coherence width of the source generating such a beam to ensure robust entanglement of the beam over a desired propagation distance in the atmosphere under any turbulence conditions. We anticipate our results to be instrumental in facilitating classical and quantum information processing through the turbulent atmosphere and in informing further studies into classical entanglement with random light.

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Data availability. Data underlying the results presented in this paper are not publicly available at this time but may be obtained from the author upon reasonable request.

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